The Curious Quest

Issue Number 1

Centre for Mathematical Outreach, MAX

"Logic is invincible because in order to combat logic it is necessary to use logic."

– Pierre Boutroux

§ Reader's Delight

Blaise Pascal – The Probability Pioneer

In 1654, a French nobleman and gambler, Antoine Gombaud (known as the Chevalier de Méré), posed a puzzle to Blaise Pascal: How should the winnings be fairly divided in an interrupted game of chance? Suppose two players are competing to reach a set number of points by rolling dice, but the game is stopped before either wins. How can the prize be distributed based on their current standings? Pascal turned to Pierre de Fermat, and their discussion led to the foundation of probability theory. They devised a method to calculate the chances of each player winning if the game continued. This problem, known as the "Problem of Points," helped establish the concept of expected value—the basis for modern probability and risk assessment.

Figure 1: Blaise Pascal

Pascal's work extended beyond probability. He developed Pascal's Triangle, crucial in combinatorics, and formulated Pascal's Principle in fluid mechanics. Later, he shifted to philosophy, writing Pensées, where he proposed Pascal's Wager—arguing that believing in God is a rational bet.

Though he died at 39, his impact on mathematics, physics, and philosophy endures. Today, probability theory influences everything from insurance to artificial intelligence, all stemming from a simple gambling question.

Grinstead, C. M., & Snell, J. L. (1997). **Introduction to Probability**. American Mathematical Society.

§ The Problem Arena

Problem 1

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that $f'(x) > g'(x) \forall x$. Then the graphs of f and g

a) Intersect exactly once

b) Intersect at most once

c) Do not intersect

d) Can intersect more than once

Problem 2

Let A_n be the area of the polygon whose vertices are given by the n-th roots of unity in the complex plane. Compute $\lim_{n\to\infty} A_n$.

Problem 3

Let f be a differentiable function such that $f(f(x)) = x$ for $x \in [0,1]$. Suppose $f(0) = 1$. Determine the value of

$$\int_0^1 (x - f(x))^{2024}\, dx$$

§ The Enigma Box

Omnitrix Correlation Conundrum

Ben Tennyson, the hero with the Omnitrix, has been pondering over a statistical pattern in his alien transformations. He believes there might be a deeper, hidden connection in how frequently he uses certain aliens. To explore this possibility, Ben enlists the help of Azmuth, the creator of the Omnitrix. Azmuth, being a brilliant scientist, explains that Ben's alien transformations can be modeled using random variables. Specifically, let's define three key transformations as random variables:

- X for XLR8, the speedster, who Ben uses for fast-paced chases and quick getaways.
- Y for Heatblast, the fiery alien, used for combat and offensive strategies.
- Z for Diamondhead, the crystalline alien, typically used for defense and durability.

Azmuth reveals that after analyzing Ben's transformation data, he finds the following correlations:

$$\rho(X,Y) = \frac{e}{\pi} \quad \text{and} \quad \rho(X,Z) = \frac{e}{\pi}$$

where $\rho(A,B)$ denotes the correlation coefficient between two random variables A and B.
Ben is puzzled by this information. "What does this tell us about the relationship between Heatblast and Diamondhead?" he asks.
Azmuth says,"That, Ben, is the real challenge."
Your Task: Based on the provided correlations, determine the maximum and minimum possible values for the correlation between Heatblast and Diamondhead, $\rho(Y,Z)$.
You know the following facts:
- $\rho(X,Y) = \frac{e}{\pi}$ and $\rho(X,Z) = \frac{e}{\pi}$, meaning that the uses of XLR8 are statistically related to both Heatblast and Diamondhead.
- The key question is how the use of Heatblast and Diamondhead are correlated, given their shared relationship with XLR8.
Can you figure out the bounds for $\rho(Y,Z)$, the correlation between Heatblast and Diamondhead? Could it be positive, negative, or even zero? What's the mathematical limit of their relationship?

§ This Issue's Contributors

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We do not claim to be the creators of any questions shared in *The Curious Quest*, unless specified otherwise.

If you have any questions, puzzles, or stories that you want to share, kindly mail them to centre.math.outreach@gmail.com!

§ Hints & Solutions - Previous Issue

Problem 1

Use the polynomial division algorithm to get that the remainder must be of the form $ax^2 + bx + c$ and that the divsor polynomial can be factored as $(x+1)(x^2+1)$. Then use complex numbers to get a closed form (it will be complicated!).

Problem 2

Let the sides be a, a, b, then the triangle will be formed iff $2a > b$. For any a, b can vary from 1 to $2a-1$ and we have the additional restriction that $a, b \leq 2026$.

(1) If $a \leq 2026/2 = 1013$, b varies from 1 to $2a-1$. Now, if $a = 1$, $b = 1$ has no choice, if $a = 2$, $b = 1, 2, 3$, if $a = 3$, $b = 1, 2, 3, 4, 5$ and if $a = 4$, $b = 1, 2, 3, 4, 5, 6, 7$; clearly the number of triangles that can be formed is equal to the sum of first 1013 odd natural numbers, which is 1013^2.

(2) If $1014 \leq a \leq 2026$, b can take any value between 1 and 2026. Thus for each a, we get 2026 triangles. Therefore total number of triangles in this case $= 1013 \times 2026$.
And hence, in total we have $1013^2 + 1013 \times 2026 = 3 \times 1013^2$ such isosceles triangles.

Problem 3

To show that the given couple is a metric space is trivial by the triangle inequality. The shortest path is not unique; for example, according to the given metric, the distance between (0,0) and (3,3) is 6 units. This distance can be charted in multiple ways: (1) Going 3 units to the right and then 3 units up. (2) Going 3 units up and then 3 units right. In fact, for any point (x, y), there are $\binom{x+y}{x}$(try proving this!) ways to get to that point from the origin that follows the given metric. And this metric is $\geq$ the Euclidean metric, equality arising only when: The x component between 2 points is identical or when the y component is identical.

A Weird Power Tower

Study the sequence defined by $s_1 = \sqrt{2}$, and $s_n = \sqrt{2}^{s_{n-1}}$ $\forall n > 1$. Show that it is increasing and bounded above by 2, which implies that it cannot be 4.