The Curious Quest

Issue Number 2

Centre for Mathematical Outreach, MAX

"In mathematics, the art of proposing a question must be held of higher value than solving it."

– Georg Cantor

§ Reader's Delight

Gödel's Dictatorship Proof

Kurt Gödel, the Austrian logician famous for his Incompleteness Theorems, was deeply paranoid. When applying for U.S. citizenship in 1947, he obsessively studied the U.S. Constitution and believed he had found a logical loophole that could allow the U.S. to become a dictatorship. His friend Albert Einstein and economist Oskar Morgenstern were alarmed, as Gödel often fixated on theoretical dangers. They accompanied him to his citizenship interview to ensure he wouldn't bring up the loophole. However, when the examiner casually mentioned, "You know, this country cannot become a dictatorship," Gödel immediately responded, "But I know a proof that it can!"

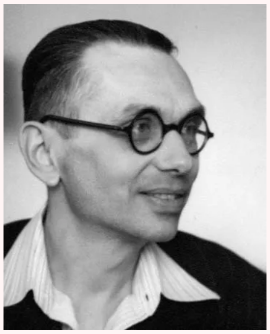

Figure 1: K. Gödel

Einstein and Morgenstern quickly changed the subject, preventing Gödel from derailing the interview. Despite this, he was granted U.S. citizenship. The exact loophole he found remains unclear, but his paranoia and extreme logical precision made this episode one of the quirkiest moments in mathematical history.

Logical Dilemmas: The Life and Work of Kurt Gödel, John W. Dawson Jr.

§ The Problem Arena

Problem 1

Let $f \in \mathbf{R}_k[x]$. If

$$\int_a^b x^i f(x)\,dx = 0 \text{ for } i = 0, 1, .., k$$

Where a and b are fixed real numbers such that $a < b$. Find all such f.

Problem 2

Find the greatest common divisor of the numbers

$$A_n = 2^{3n} + 3^{6n+2} + 5^{6n+2}$$

for $n = 0, 1, \ldots, 2025$

Problem 3

Given any $n \in \mathbb{N}$, Let a_n denote the number of binary strings of length n that do not have consecutive 1s.
Does the series $\sum_{k=0}^{\infty} \frac{a_k}{k!}$ converge? If yes, compute it.

§ The Enigma Box

Delay, Destiny, and the Doom of Planet X

Kakarot and Saitama have been assigned separate missions on Planet X and are scheduled to arrive at the same location at a specified time. Each of them will arrive with a delay ranging from 0 to 80 minutes, with all possible pairs of delays being equally likely. Both of them will do their work for 25 minutes before leaving.
If they meet, their combined presence will awaken an ancient cosmic being that instantly destroys Planet X. However, if they don't meet, a secret time anomaly will teleport one of them 100 years into the past. If one of them is sent to the past, there is a 30% chance that they will interfere with history in a way that destroys Planet X. What is the probability that Planet X gets destroyed?

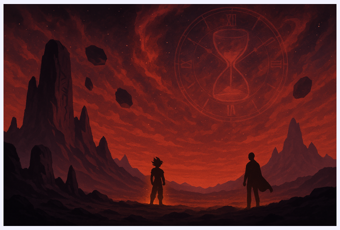

Figure 2: Planet X

§ This Issue's Contributors

- **Problem 1**: *Aadi Upraity, St. Xavier's College, Mumbai*
- **Problem 2**: *Aadi Upraity, St. Xavier's College, Mumbai*
- **Problem 3**: *Aadi Upraity, St. Xavier's College, Mumbai*
- **Enigma Box**: *Aadi Upraity, St. Xavier's College, Mumbai*
- **Reader's Delight**: *Harsh Nagda, St. Xavier's College*

If you have any questions, puzzles, or stories that you want to share, kindly mail them to centre.math.outreach@gmail.com!

§ Hints & Solutions - Previous Issue

Problem 1

Define $h(x) = f(x) - g(x)$, since the RHS is differentiable, h is also differentiable. Then $h'(x) = f'(x) - g'(x) > 0$ for all real x. Which implies that h is a strictly increasing function. So h can intersect with the x axis **at most** once, that is $h(x) = 0$ for maximum one value, which means that $f(x) = g(x)$ has either no solution of only 1 solution, so option (b) was the correct choice.

Problem 2

Join each root to the origin to get n many triangles in the polygon. Now the area of each of these triangles can be given by the formula

$$\frac{1}{2}ab\sin(\theta)$$

Where a, b are 2 side lengths and θ is the angle made by the same two sides of the triangle. Let T_n denote the area of any triangle formed by 2 consecutive roots of unity and origin:

$$T_n = \frac{1}{2}|z_k||z_{k+1}|\sin\left(\frac{2\pi}{n}\right)$$

Since $|z_k| = 1$ for all k,

$$T_n = \frac{1}{2}\sin\left(\frac{2\pi}{n}\right)$$

It's clear that $A_n = n \times T_n$, thus

$$\begin{aligned}
\lim_{n\to\infty} A_n &= \lim_{n\to\infty} \frac{n}{2}\sin\left(\frac{2\pi}{n}\right) \\
&= \pi \lim_{n\to\infty} \frac{\sin\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)} \\
\lim_{n\to\infty} A_n &= \boxed{\pi}
\end{aligned}$$

Problem 3

$$f(f(x)) = x$$

$$f(0) = 1 \implies f(1) = 0$$

$$I = \int_0^1 (x - f(x))^{2024}\,dx \tag{1}$$

Substitute $x = f(t) \implies f(x) = t$

$$x = f(t) \implies dx = f'(t)dt$$

$$x : 0 \to 1 \implies t = f(x) : 1 \to 0$$

$$I = \int_1^0 (f(t) - t)^{2024} f'(t)dt = -\int_0^1 (t - f(t))^{2024} f'(t)dt$$

$$I = -\int_0^1 (x - f(x))^{2024} f'(x)dx \tag{2}$$

Adding (1) and (2):

$$2I = \int_0^1 (x - f(x))^{2024}(1 - f'(x))dx$$

A simple $u = x - f(x)$ substitution leads us to

$$I = \frac{1}{2025}$$

Source: ISI BStat entrance, UGB, 2016.

Omnitrix Correlation Conundrum

This problem can be solved very easily if you know that the correlation coefficient between two vectors is simply the cosine of the angle between them. So we consider the given r.v.'s as $\mathbf{x}, \mathbf{y}, \mathbf{z}$ vectors. Hence,

$$\rho(\mathbf{x}, \mathbf{y}) = \cos\theta = e/\pi$$

$$\rho(\mathbf{x}, \mathbf{z}) = \cos\theta = e/\pi$$

Let the angle between vectors $\mathbf{y}$ and $\mathbf{z}$ be α. Then, $\rho(\mathbf{y}, \mathbf{z}) = \cos\alpha$ Since cosine is a strictly decreasing function in $[0,\pi]$, we only need to maximize/minimize α to obatain the minimum/maximum correlation respectively.

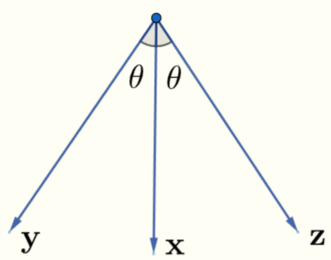

Figure 3: Vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$

In this case, the maximum angle is $\alpha = 2\theta$ and the minimum angle is 0. For minimum, by simple trigonometric identities, we have:

$$\begin{aligned}\cos(\alpha) &= \cos 2\theta \\ &= \cos^2\theta - \sin^2\theta \\ &= (e/\pi)^2 - (1 - (e/\pi)^2) \\ &= 2(e/\pi)^2 - 1 \\ &\approx 0.4973\end{aligned}$$

And the maximum correlation is achieved when the vectors coincide, $\cos 0 = 1$. In conclusion,

$$2(e/\pi)^2 - 1\,(\approx 0.4973) \leq \rho(\mathbf{y}, \mathbf{z}) \leq 1$$