The Curious Quest

Issue Number 3

Centre for Mathematical Outreach, MAX

"Mathematics reveals its secrets only to those who approach it with pure love, for its own beauty."

– Archimedes

§ Reader's Delight

Charting infinity at the cost of sanity

In the quiet corridors of 19th-century mathematics, a revolutionary idea began to take shape in the mind of Georg Cantor. While most mathematicians feared infinity as an untouchable concept, Cantor dared to ask: Are all infinities the same?

Cantor discovered something astonishing infinities could have different sizes! Using his now-famous diagonalization argument, he proved that while the set of natural numbers is infinite, the set of real numbers is an even "larger" infinity. This was not mere philosophy; it was rigorous mathematics.

However, Cantor's journey was far from smooth. His radical ideas were met with fierce resistance. The formidable Leopold Kronecker, a prominent mathematician of the time, dismissed Cantor's work, claiming, "God made the integers; all else is the work of man." The academic backlash took a toll on Cantor's mental health.

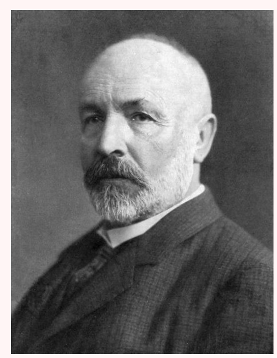

Figure 1: G. Cantor

Yet, Cantor remained steadfast. He once wrote, "The essence of mathematics lies in its freedom." Today, his work forms the foundation of modern set theory, and his courage reminds us that even in the face of adversity, daring to think differently can reshape the entire landscape of knowledge.

Dauben, J. W. (1979). *Georg Cantor: His Mathematics and Philosophy of the Infinite.* Princeton University Press., Bell, E. T. (1937). *Men of Mathematics.* Simon & Schuster.

§ The Problem Arena

Problem 1

Determine all pairs (a, b) of real numbers that maximize:

$$\int_a^b e^{\cos x}(-x^2 - 5x + 500)$$

Problem 2

Define 3 functions – f, d and g:

$$f(x) = \sin^{19}(x) - \cos^{26}(x)$$

$$d(p,q) = \frac{|p-q|}{1+|p-q|}$$

$$g(x) = d(489x, x^2 + 28980)$$

Find the number of solutions to the equation $f(x) = \operatorname{sgn}(g(x))$ in the range $[-22\pi, 22\pi]$. Where,

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Problem 3

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying $f(1) = k$ and $f(x) + \int_0^x t f(t)\,dt + x^2 = 0 \quad \forall x \in \mathbb{R}$. Then,

a) Has maxima at $x = 1$ if $k = -1$

b) f is an odd function

c) $g(x) = f(x) + 2$ can be a probability density function for an appropriate value of k.

d) $\lim_{n\to\infty} f(x) = -2$

§ The Enigma Box

When Divergence met Determination

A colossal tortoise has its shell tethered to a post with a massive elastic cord. A tiny ant sits on the post, watching the tortoise intently. Suddenly, the tortoise notices the ant, surges forward, and moves exactly one kilometer away from the post while still being connected by the elastic cord. The ant, undeterred, leaps onto the stretched cord, landing just one centimeter from the post. The tortoise, spotting this, advances another kilometer, making it a total of two kilometers from the post. The ant remains persistent, hopping another centimeter along the stretched cord. This pattern continues, with the tortoise moving another kilometer each time and the ant inching forward by a centimeter along the cord. Will the ant ever reach the tortoise? (Assume an infinite, flat landscape.)

§ This Issue's Contributors

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If you have any questions, puzzles, or stories that you want to share, kindly mail them to centre.math.outreach@gmail.com!

§ Hints & Solutions - Previous Issue

Problem 1

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$. Then from the given equality,

$$\int_a^b f(x)\,dx = \int_a^b x f(x)\,dx = \int_a^b x^2 f(x)\,dx = \cdots = \int_a^b x^k f(x)\,dx = 0$$

Multiplying each equality with the respective a_i we get:

$$\int_a^b a_0 f(x)\,dx = \int_a^b a_1 x f(x)\,dx = \int_a^b a_2 x^2 f(x)\,dx = \cdots = \int_a^b a_k x^k f(x)\,dx = 0$$

Summing each equality,

$$\int_a^b f(x) f(x)\,dx = 0 \implies \int_a^b (f(x))^2\,dx = 0 \implies \boxed{f(x) = 0.}$$

Problem 2

This was a problem from JBMO, 1999.

We begin with simple ones, A_0 and A_1. $A_0 = 35$, thus the GCD *must* be either 1,5,7 or 35. For $A_1 = 397194 = 2 \cdot 3 \cdot 7^3$, we infer that the GCD now must be either 1 or 7. Let see $A_n \mod 7$ for all n.
Breaking down it's components,

$$\begin{aligned} 2^3 \equiv 8 \equiv 1 \mod 7 &\implies 2^{3n} \equiv 1 \mod 7 \\ 3^6 \equiv 1 \mod 7 &\implies 3^{6n+2} \equiv 2 \mod 7 \quad (1) \\ 5^6 \equiv 1 \mod 7 &\implies 5^{6n+2} \equiv 4 \mod 7 \quad (2) \end{aligned}$$

Where the left hand of (1) and (2) is due to Fermat's Little Theorem and right hand member is due to $9 \equiv 2 \mod 7$, $25 \equiv 4 \mod 7$. And thus

$$A_n \equiv 0 \mod 7 \quad \forall n \in \mathbb{N}$$

Hence $\text{GCD}(A_0, A_1, \ldots, A_{2025}) = 7$.

Problem 3

Let a_n be the number of binary strings of length n with no consecutive 1s. We have the recurrence

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 2,$$

with initial conditions $a_0 = 1$, $a_1 = 2$. [Why? to construct any n digit binary string, we have two choices for the right most digit 0 and 1. If we have 0 as the right most digit then we dont care what appears in the remaining $n-1$ digits, so the number of strings we get by this case is equal to a_{n-1}. The second case is: 1 as the right most digit, due to the condition given in the statement, it follows that the $(n-1)$th digit must be 0, and again, we dont care what appears in the remaining $n-2$

digits, thus from this case we have a_{n-2} number of strings. Now there cases are exclusive and exhaustive, thus by addition principle, the result follows.]
By induction we can easily prove:

$$a_n = F_{n+2},$$

where $\{F_n\}$ is the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$.
Consider the series

$$S = \sum_{k=0}^{\infty} \frac{a_k}{k!} = \sum_{k=0}^{\infty} \frac{F_{k+2}}{k!}.$$

Using the Binet formula,

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}, \quad \text{where } \phi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2},$$

we rewrite

$$S = \sum_{k=0}^{\infty} \frac{1}{\sqrt{5}} \frac{\phi^{k+2} - \psi^{k+2}}{k!} = \frac{1}{\sqrt{5}} \left(\phi^2 e^{\phi} - \psi^2 e^{\psi}\right).$$

Which follows due to the definition of e's MacLaurin Series.

$$\boxed{\sum_{k=0}^{\infty} \frac{a_k}{k!} = \frac{1}{\sqrt{5}} \left(\phi^2 e^{\phi} - \psi^2 e^{\psi}\right).}$$

Delay, Destiny, and the Doom of Planet X

Let $X, Y \sim \text{Uniform}[0, 80]$ be independent random variables representing the arrival delays (in minutes) of Kakarot and Saitama. Both stay for 25 minutes. They meet if and only if their time intervals overlap:

$$|X - Y| \leq 25.$$

Since (X, Y) is uniform over the square $[0, 80]^2$ of area 6400, the probability of meeting equals the area of the region where $|X - Y| \leq 25$ divided by 6400.
The set $\{(X, Y) : |X - Y| > 25\}$ consists of two right triangles outside the band around the diagonal. Each triangle has legs of length $80 - 25 = 55$, so area

$$\frac{55^2}{2} = \frac{3025}{2} = 1512.5.$$

Thus, the area where they meet is

$$6400 - 2 \times 1512.5 = 6400 - 3025 = 3375.$$

Therefore,

$$P(\text{meet}) = \frac{3375}{6400} = \frac{135}{256}.$$

$$P(\text{no meet}) = 1 - \frac{135}{256} = \frac{121}{256}.$$

If they meet, Planet X is destroyed immediately. If they do not meet, one is teleported to the past, with a 30% chance to destroy Planet X.
Hence, the total probability Planet X is destroyed is

$$P(\text{destroyed}) = P(\text{meet}) + P(\text{no meet}) \times 0.3 = \frac{135}{256} + \frac{121}{256} \times 0.3 = \frac{135}{256} + \frac{36.3}{256} = \frac{171.3}{256} \approx 0.669.$$