The Curious Quest

Issue Number 5

Centre for Mathematical Outreach, MAX

"I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives."

– Charles Hermite

§ Reader's Delight

The Boy Who Summed Faster Than You Can Say Arithmetic

Figure 1: Carl Friedrich Gauss

When Carl Friedrich Gauss was just a young schoolboy—about 8 or 9 years old—his teacher gave the class what he thought was a nice, long task to keep them busy: "Add all the numbers from 1 to 100." While the rest of the class began slowly writing 1 + 2 + 3 + 4 + ... Gauss famously paused, thought for a moment, and then wrote down a single number on his slate: 5050

The teacher was stunned. What Did Gauss See? Gauss realized that the numbers could be paired: (1 + 100) + (2 + 99) + (3 + 98) + ... + (50 + 51). Each pair sums to 101, and there are 50 such pairs: 50 × 101 = 5050. This brilliant insight led to the famous formula: 1 + 2 + 3 + ... + n = n(n + 1)/2 .

While the exact details may have been polished over time, the story of young Gauss summing numbers from 1 to 100 captures a deeper truth: mathematics rewards insight over effort, and even simple problems can reveal elegant patterns to those who look closely.

Bell, E. T. Men of Mathematics. Simon and Schuster, 1937. Chapter 1: "The Prince of Mathematicians".

§ The Problem Arena

Problem 1

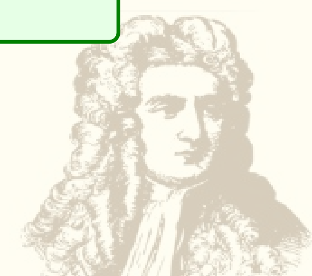
Let $Q(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with integer coefficients, and $0 \le a_i < 3$ for all $0 \le i \le n$. Given that $Q(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $Q(2)$.

Problem 2

Let $f(x)$ be a polynomial in x with integer coefficients and suppose that for 2025 distinct integers $a_1, \ldots, a_{2025}$, one has

$$f(a_1) = f(a_2) = \cdots = f(a_{2025}) = 2.$$

Show that there does not exist an integer b such that $f(b) = 9$.



Problem 3

Define a sequence $\{O_n\}_{n=0}^{\infty}$ by $O_0 = O_1 = O_2 = 1$, and thereafter by the condition that

$$O_n O_{n+3} - O_{n+1} O_{n+2} = n! \quad \text{for all } n \geq 0.$$

Let α denote the exponent of 2 in the prime factorization of O_{2025}. Find the value of α.

§ The Enigma Box

Sherlock, Moriarty, and the Prison of Parity

In a showdown of intellect, Dr. Strange captures both Sherlock Holmes and Professor Moriarty, challenging them with a logic-based escape test inside a magical prison containing 100 cells, each numbered from 1 to 100. Each cell door is initially closed. Dr. Strange announces the following procedure:

"There are 100 passes. On the 1st pass, every cell door is opened. On the 2nd pass, every second cell door (2, 4, 6, ...) is toggled — opened if closed, or closed if opened. On the 3rd pass, every third cell door is toggled. This continues up to the 100th pass, where only the 100th door is toggled." Dr. Strange smiles and adds:

"At the end of the 100th pass, you are to identify exactly which doors remain open. If either of you answers correctly, you go free. If not, you remain in the Parity Prison forever!" Sherlock immediately replies with the correct answer, while Moriarty remains silent.

What did Sherlock answer and why?

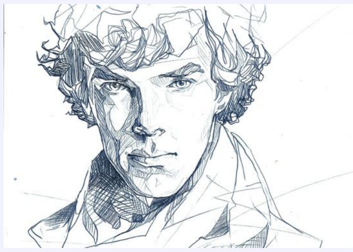


Figure 2: William Sherlock Scott Holmes

§ This Issue's Contributors

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If you have any questions, puzzles, or stories that you want to share, kindly mail them to centre.math.outreach@gmail.com!

§ Hints & Solutions - Previous Issue

Problem 1

Transform S_n as

$$S_n = \frac{1}{n^2}\sum_{k=1}^{n} k\log k - \frac{1}{2}\log n$$
$$= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\left(\log\frac{k}{n} + \log n\right)\right) - \frac{1}{2}\log n$$
$$= \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n} + \frac{\log n}{n^2}\sum_{k=1}^{n} k - \frac{1}{2}\log n$$
$$= \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n} + \log n\cdot\frac{n+1}{2n} - \frac{1}{2}\log n$$
$$= \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n} + \frac{\log n}{2n}$$

Here the last term $\frac{\log n}{2n}$ converges to 0. The sum $\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n}$ is a Riemann sum for the integrable function $f(x) = x\log x$ on the segment $[0,1]$ with the uniform grid

$$\left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}.$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\log\frac{k}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 x\log x\,dx$$
$$= \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_0^1 = -\frac{1}{4}$$

Hence, $\lim S_n$ exists, and

$$\lim S_n = -\frac{1}{4}.$$

Problem 2

Without the loss of generality, let $\mathrm{X}_1 = max(X_1, X_2, ..., X_k)$.

Then using simple sandwich theorem, prove that the limit is equal to X_1.

Problem 3

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Doom Invariance

Let n be the number of Infinity Stones and let $E(n)$ denote the total energy extracted. Each time Dr. Doom splits a pile of $a+b=n$ stones, he gains ab energy and continues recursively on each subgroup until all piles have size 1.
We observe that:

$$E(n) = ab + E(a) + E(b)$$

We claim that:

$$E(n) = \frac{n(n-1)}{2}$$

Base Cases: For $n=2$, the only split is $1+1$, giving 1 unit of energy: $\frac{2(2-1)}{2} = 1$. For $n=3$, we can split as $1+2$ giving $2+1=3$ units: $\frac{3(3-1)}{2} = 3$.
Inductive Step: Assume $E(k) = \frac{k(k-1)}{2}$ for all $k<n$. Now split n into x and $n-x$. Then,

$$E(n) = x(n-x) + \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2}$$

A quick simplification shows this equals $\frac{n(n-1)}{2}$, completing the inductive proof.
Final Answer:

$$E(2025) = \frac{2025 \cdot 2024}{2} = 2025 \cdot 1012$$