The Curious Quest

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"There are only two kinds of math books: Those you cannot read beyond the first sentence, and those you cannot read beyond the first page"

– Chen-Ning Yang

§ Reader's Delight

The Last Theorem of Évariste Galois

Figure 1: Évariste Galois

Évariste Galois was a mathematical firebrand born in 1811 near Paris, a time when France simmered with revolution. By 17, he'd cracked a problem that had stumped scholars for centuries: why some polynomial equations could be solved algebraically and others couldn't. His creation "group theory" became a pillar of modern mathematics, unlocking the secrets of symmetry in everything from crystals to quantum physics. But Galois wasn't just a thinker; he was a fighter. A staunch republican, he joined protests against the monarchy, got expelled from school, and even tried to enlist in the National Guard only to be rejected. His temper landed him in prison twice, once for toasting the king with a knife in hand. Then came 1832. At 20, he faced dual rumors swirling over a woman, Stéphanie-Félicie, or a political grudge. The night before, he didn't sleep. Instead, he poured his soul into a letter to a friend, frantically scribbling his mathematical discoveries, including the foundations of what's now called Galois Theory. Shot in the stomach the next morning, he lingered in agony before dying the following day, May 31, 1832. His brother and friends preserved his chaotic notes, which mathematicians later deciphered, revealing a genius snuffed out too soon. Galois showed that brilliance and rebellion can ignite a legacy even if it ends in blood.

Stewart, I. (2008). *Why Beauty Is Truth: A History of Symmetry.*

§ The Problem Arena

Problem 1

Let $p(x)$ be a polynomial of degree k. For $n \in \mathbb{N}$, define $a_n = p(n+1) - p(n)$. If $\{a_n\}_{n=1}^{\infty}$ is an arithmetic progression, then:

a) $k \leq 1$ b) $k \leq 2$

c) $k \geq 3$ d) $k \geq 4$

Problem 2

Let $f : [0,1] \to [0,1]$ be a function such that:

$$|f(x) - f(y)| < |x - y|$$

for all $x \neq y \in [0,1]$. Which of the following statements is/are true?

a) There exists at least one point $z \in [0,1]$ such that $f(z) = z$

b) There exists at most one point $z \in [0,1]$ such that $f(z) = z$

Problem 3

(This is a famous problem) Let α and β be the roots of the quadratic equation $x^2 + mx - 1 = 0$ where m is an odd integer. Let $\gamma_n = \alpha^n + \beta^n$ for $n \geq 0$ Prove the following statements:

a) $\gamma_n \in \mathbb{Z}$ for all $n \geq 0$

b) $\gcd(\gamma_n, \gamma_{n+1}) = 1$ for all $n \geq 0$

§ The Enigma Box

Flipping out

Backstage after a sold-out Live Aid Redux concert, Freddie Mercury and David Bowie decide to spice things up with a little game, because why not.
Bowie grabs 1990 fair coins, while Freddie, grabs with 1991 coins (because he is a bit extra). With the crowd watching, they both flip all their coins.
The question floating around the green room is:
What's the Probability that Freddie gets strictly more heads than Bowie.
And a nerd starts solving this...

Figure 2: IYKYK

§ This Issue's Contributors

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We do not claim to be the creators of any questions shared in *The Curious Quest*, unless specified otherwise.

If you have any questions, puzzles, or stories that you want to share, kindly mail them to `centre.math.outreach@gmail.com`!

§ Hints & Solutions - Previous Issue

Problem 1

Given that f is one-one, and continuous, f is either strictly increasing or strictly decreasing.
Case 1) f is strictly increasing.
Let $f^n(x_0) = x_0$ WLOG, let $f(x_0) > x_0 \implies f^2(x_0) > f(x_0) > x_0$ and so on, i.e. $f^k(x_0) > x_0$ for all $k \in \mathbb{N}$ (By induction). Which is a contradiction, since we know that there is at least one n such that $f^n(x_0) = x_0$ the case where $f(x_0) < x_0$ is similarly proven. So, $f(x) = x$ is f is increasing
Case 2) Let f be decreasing.
Case 2.1) Let n be even. WLOG, let $f^2(x_0) > x_0 \implies f^3(x_0) < f(x_0) \implies f^4(x_0) > f^2(x_0) > x_0$ and so on. Using induction we can prove that $f^{2k}(x_0) > x_0$ for all $k \in \mathbb{N}$. A contradiction. Analogous result holds for $f^2(x_0) < x_0$.
Case 2.2) We leave this to you.

Problem 2

Multiply equation (2) by 27 and add to equation (1):

$$x^3 + 9x^2y + 27y^3 + 27xy^2 = 64$$

Factor as:

$$(x + 3y)^3 = 64 = 4^3$$

Therefore: $x + 3y = 4$
Substituting $x = 4 - 3y$ into equation (2):

$$y^3 - 2y^2 + 1 = 0$$

$$(y - 1)(y^2 - y - 1) = 0$$

Solutions:

$$(x, y) = (1, 1) \tag{1}$$

$$(x, y) = \left(\frac{5 - 3\sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}\right) \tag{2}$$

$$(x, y) = \left(\frac{5 + 3\sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}\right) \tag{3}$$

Problem 3

Three points are chosen uniformly at random on a circle with circumference 12. Let D be the minimum pairwise circular distance, where the distance between points $x, y \in [0, 12)$ is $\min(|x - y|, 12 - |x - y|)$. We need the CDF of D, $P(D > 2)$, and $E[D]$.
1. CDF of D: The CDF is $F_D(d) = P(D \leq d) = 1 - P(D > d)$, where $D = \min(\text{dist}(X_1, X_2), \text{dist}(X_2, X_3), \text{dist}(X_1, X_3))$. The three points divide the circle into three arcs A_1, A_2, A_3 with $A_1 + A_2 + A_3 = 12$. The minimum distance relates to the arcs, considering circular distances: $D = \min(A_1, A_2, \min(A_1 + A_2, A_3))$.
To simplify, normalize the circumference to 1 (so scale distances by 1/12). For three points on a unit circle, the probability that all pairwise distances exceed $\delta = d/12$ is known to be $(1 - 3\delta)^2$ for $0 \leq \delta \leq 1/3$ (from geometric probability on a circle). Scaling back ($\delta = d/12$, $\delta \leq 1/3 \implies d \leq 4$):

$$P(D > d) = \left(1 - \frac{d}{4}\right)^2, \quad 0 \le d \le 4$$

$$F_D(d) = 1 - \left(1 - \frac{d}{4}\right)^2 = \frac{d}{2} - \frac{d^2}{16}, \quad 0 \le d \le 4$$

For $d > 4$, $P(D > d) = 0$ since the sum of two arcs plus the third exceeds 12, making $D > 4$ impossible. Thus:

$$F_D(d) = \begin{cases} 0 & \text{if } d < 0 \\ \frac{d}{2} - \frac{d^2}{16} & \text{if } 0 \le d \le 4 \\ 1 & \text{if } d > 4 \end{cases}$$

2. $P(D > 2)$**:** Using the CDF:

$$P(D > 2) = 1 - F_D(2) = 1 - \left(\frac{2}{2} - \frac{2^2}{16}\right) = 1 - \left(1 - \frac{1}{4}\right) = \frac{1}{4}$$

Alternatively, $P(D > 2) = \left(1 - \frac{2}{4}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. This confirms the CDF's consistency.

3. $E[D]$**:** For a non-negative random variable, $E[D] = \int_0^\infty P(D > t)\, dt$. Since $P(D > t) = 0$ for $t > 4$:

$$E[D] = \int_0^4 \left(1 - \frac{t}{4}\right)^2 dt$$

we get,

$$E[D] = \frac{4}{3}$$