The Curious Quest

Issue Number 9

Centre for Mathematical Outreach, MAX

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

– John von Neumann

§ Reader's Delight

The Last Man Who Knew Everything

Figure 1: John von Neumann

Polymaths – individuals with deep expertise across multiple, seemingly unrelated fields have always been rare, and in the modern age of increasing specialization, they've become even scarcer. The vast expansion of knowledge in every discipline has made it nearly impossible for one person to master multiple domains at the cutting edge. Yet John von Neumann (1903–1957) was perhaps the last true polymath of the 20th century, a mind whose breadth and depth of understanding spanned mathematics, physics, economics, computer science, engineering, and even military strategy.

A child prodigy with a legendary memory, he earned a Ph.D. in mathematics by 23 and went on to leave lasting marks in almost every field he touched. He formalized the mathematics of quantum mechanics, co-founded game theory with Oskar Morgenstern, played a crucial role in the Manhattan Project by developing models for nuclear implosion, and invented the von Neumann architecture, the blueprint for nearly all modern computers. His peers—great scientists in their own right—often spoke of him in awe; Edward Teller called von Neumann "the only truly brilliant man" he had ever known.

In a well-known incident, a friend challenged von Neumann by giving him a complex problem involving trains and infinite summations. Before the friend could finish describing the "trick" solution (using a geometric series), von Neumann immediately blurted out the correct answer. When asked how he did it so fast, von Neumann simply said, *"There are two ways to do this problem, and I did it both ways."* Von Neumann's influence permeates multiple disciplines, and his legacy endures as one of the greatest minds of the 20th century – and perhaps the last person to truly know it *all*.

Dyson, George. "The Legacy of John von Neumann." Scientific American, vol. 272, no. 6, June 1995, pp. 90–95.

§ The Problem Arena

Problem 1

Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the relation

$$4f(3-x) + 3f(x) = x^2$$

for any real x. We get that $\lceil f(69) - f(25) \rceil = z^2$. Find z.

Problem 2

Determine all positive integers n for which there exist positive integers a, b, and c satisfying

$$2a^n + 3b^n = 4c^n.$$

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Problem 3

Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

§ The Enigma Box

3's a Curse: Dumbledore's Card Game

At Hogwarts, Professor Dumbledore has devised a magical challenge involving his enchanted stash of chocolate frogs. Each frog comes with a collectible card featuring one of n different famous witches or wizards, chosen at random with equal probability. A student—perhaps Harry, Hermione, or Ron—is trying to collect all n different cards. However, there's a magical twist: if the student receives the same card **three times**, it vanishes permanently from the collection and can no longer be obtained. Dumbledore, ever curious about probability, wonders: what is the *expected number of chocolate frogs* the student needs to buy in order to collect **all** n unique cards, given this vanishing rule?

Let $E(n)$ represent this expected number. How does this change from the classical coupon collector's expectation of

$$n\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)?$$

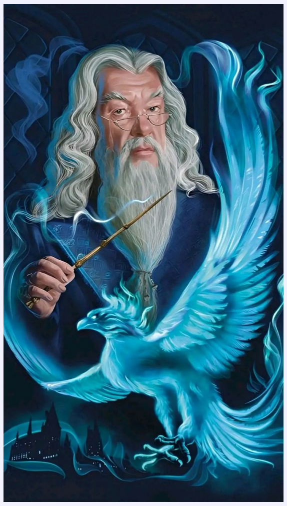

Figure 2: Albus Dumbledore

Can a recursive strategy be used to compute $E(n)$ for small values like $n = 2, 3, 4$? And as a further twist, if Fred and George Weasley decide to play the game together by pooling their cards and avoiding repeats, can teamwork lower the expected number of frogs needed to complete the set?

§ This Issue's Contributors

- **Problem 1**: *Ayush Mishra, St. Xavier's College, Mumbai*

- **Problem 2**: *Ayush Mishra, St. Xavier's College, Mumbai*
- **Problem 3**: *Ayush Mishra, St. Xavier's College, Mumbai*
- **Enigma Box**: *Ayush Mishra, St. Xavier's College, Mumbai*
- **Reader's Delight**: *Ayush Mishra, St. Xavier's College, Mumbai*

We do not claim to be the creators of any questions shared in *The Curious Quest*, unless specified otherwise.

If you have any questions, puzzles, or stories that you want to share, kindly mail them to centre.math.outreach@gmail.com!

§ Hints & Solutions - Previous Issue

Problem 1

We are asked to count the number of permutations σ of the set $\{0, 1, 2, \ldots, 9\}$ such that for every position $i > 0$, the value $\sigma(i)$ is within 1 of some earlier number $\sigma(j)$ for $j < i$. Formally, for each $i \neq 0$, there exists $j < i$ such that

$$|\sigma(i) - \sigma(j)| = 1.$$

This condition implies that the numbers must be added in such a way that each new number is adjacent to an existing one (by absolute difference 1). This is equivalent to constructing a permutation where we begin with any one number and iteratively insert numbers adjacent (in value) to those already placed.
This problem is well-known and is connected to a variant of the *restricted permutations* or *connected adjacency graphs*. For the case of the set $\{0, 1, 2, \ldots, n-1\}$, the number of such permutations is given by the number of **restricted growth strings** or valid **adjacent walks**.
For $n = 10$, the total number of such permutations is:

$$\boxed{512}$$

This result can be derived recursively or found in **OEIS sequence [A001047]** https://oeis.org/A001047, which counts the number of such valid permutations under adjacency constraints.

Problem 2

Given:
Let $\{r_1, r_2, \dots\}$ be an enumeration of all rationals in $[0, 1]$. Define, for each $n \in \mathbb{N}$:

$$f_n(x) = \begin{cases} e^x & \text{if } x \in \{r_1, r_2, \dots, r_n\} \\ e^{1-x} & \text{otherwise} \end{cases}$$

Step 1: Each f_n differs from the continuous function e^{1-x} at only finitely many points (measure zero).
$\Rightarrow f_n$ is Riemann integrable for all n.
Step 2: Pointwise limit:

$$\lim_{n\to\infty} f_n(x) = \begin{cases} e^x & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ e^{1-x} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

This function is discontinuous at every point in $[0, 1]$
$\Rightarrow$ not Riemann integrable.

$$\boxed{\text{Answer: (a),(c)}}$$

The sequence $\{f_n\}$ is pointwise convergent, but the limit function is **not** Riemann integrable on $[0, 1]$.

Problem 3

Each term is a sum of distinct powers of 4, so we can represent each number using a binary mask — a binary number where each bit indicates whether the corresponding power of 4 is included (1) or not (0), starting from at the right.

$$1280 = 4^5 + 4^4 \Rightarrow mask = 110000$$

Interpreting this binary mask as a binary number:

$$110000_2 = 48$$

Since the sequence is ordered by increasing such sums, this corresponds to the 49th number (as counting starts from 0).

Pingo

Paul reveals cards one by one from a shuffled 52-card deck with 26 red and 26 black cards. Ringo may interrupt exactly once to bet \$1 that the *next* card will be red. If he never interrupts, he is forced to bet on the final card.
Ringo's optimal strategy is to bet at the first moment when the number of red cards remaining exceeds the number of black cards remaining. That is, he waits until $r > b$, where r and b are the counts of red and black cards left in the undealt portion. If this never happens, he bets on the last card.
This strategy gives a winning probability of

$$\boxed{\frac{27}{52}} \approx 51.92\%.$$

This is slightly better than random guessing, which yields a 50% chance, and gives an edge of

$$\frac{1}{52} \approx 1.92\%.$$