The Curious Quest

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Centre for Mathematical Outreach, MAX

"The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite.."

– George Canter

§ Reader's Delight

Hilbert's Hotel: The Infinity Inn That's Always Full Yet Never Full

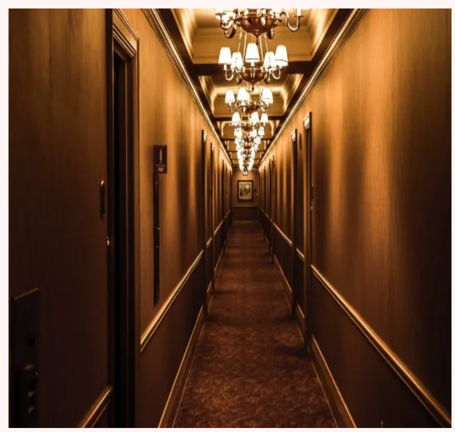

Figure 1: Hilbert's Grand Hotel

Imagine a hotel with infinitely many rooms, numbered 1, 2, 3, and so on. Every room is occupied. You walk in and ask for a bed. The manager smiles and says, "Of course, we have space." He asks every guest to move up one room. The person in room 1 moves to room 2, 2 to 3, 3 to 4, and so on. Suddenly, room 1 is free. A hotel that was completely full just made space for you.

This is Hilbert's Hotel, a thought experiment created by mathematician David Hilbert to show how strange infinity really is. Add one guest and the hotel fits them. Add infinitely many new guests and it still fits them. Just move every guest from room n to room $2n$, which frees up all the odd-numbered rooms for the newcomers.

It seems absurd, but this idea captures something deep. Infinite sets do not behave like finite ones. In a normal hotel, full means no more room. In Hilbert's Hotel, full means you just need a clever rearrangement.

This is not just a math trick. It plays a role in set theory, theoretical computer science, and even cosmology. Infinity is not a number you can touch. It is a concept that stretches logic, often feeling more like a puzzle than arithmetic. Hilbert's Hotel invites us to see infinity not as unreachable, but as curiously flexible.

Gamow, G. (1947). **One, Two, Three... Infinity: Facts and Speculations of Science**. Viking Press.

§ The Problem Arena

Problem 1

Let T be the set of all infinite sequences of the form $\{a_n\}_{n=1}^{\infty}$, where each $a_n \in \{0, 1\}$ and the sequence contains only finitely many 1's. Prove that T is countable.

Problem 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\lim_{x\to\infty} f(x) = L \quad \text{and} \quad \lim_{x\to-\infty} f(x) = M,$$

where $L, M \in \mathbb{R}$. Prove or disprove: f must attain both its supremum and infimum on $\mathbb{R}$.

Problem 3

Let X_n denote the number of heads obtained from n independent tosses of a biased coin,where the probability of heads on each toss is $p \in (0, 1)$. Let $p_n = \mathbb{P}(X_n$ be even), i.e the probability that X_n is an even no

1. Show that sequence (p_n) satisfies the recurrence relation

$$p_{n+1} = (1 - 2p)p_n + p$$

2. Show that distribution becomes fair,as $n \to \infty$ [Do try this ,with (a)]

§ The Enigma Box

The Half Distance Dilemma

Two agents, A and B, are ordered to reach a control panel exactly 1 kilometer away from their respective starting points at opposite ends of a corridor. However, they are under strict protocol: at each time step, they are allowed to travel only half the remaining distance to the control panel.
Agent A has advanced computational boots that allow them to complete infinitely many such steps in exactly 1 hour. Agent B uses conventional gear, taking 1 minute per step, regardless of the decreasing distances.
If both agents reach the control panel at the exact same time, the system will interpret it as a breach and trigger a self-destruction sequence. However, if only one reaches it, the system activates a 30-minute lockdown and resets safely.
Assume the entire process starts at time t=0.
What is the probability that the system self-destructs?
Can Agent B ever actually reach the panel?

§ This Issue's Contributors

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We do not claim to be the creators of any questions shared in *The Curious Quest*, unless specified otherwise.

If you have any questions, puzzles, or stories that you want to share, kindly mail them to centre.math.outreach@gmail.com!

§ Hints & Solutions - Previous Issue

Problem 1

Substitute $3 - x$ in place of x and simplify. You will get a system of two equations in two variables :$f(x)$ and $f(3 - x)$. Solve to get $f(x)$.

Problem 2

The answer is $n = 1$. When $n = 1$, $(a, b, c) = (1, 2, 2)$ is a solution to the given equation. We claim that there are no solutions when $n \geq 2$.

For $n = 2$, suppose that we have a solution to $2a^2 + 3b^2 = 4c^2$ with $a, b, c \in \mathbb{N}$. By dividing each of a, b, c by $\gcd(a, b, c)$, we obtain another solution; thus we can assume that $\gcd(a, b, c) = 1$. Note that we have

$$a^2 + c^2 \equiv 0 \pmod 3,$$

and that only 0 and 1 are perfect squares mod 3; thus we must have $a^2 = c^2 \equiv 0 \pmod 3$. But then a, c are both multiples of 3; it follows from

$$b^2 = \frac{12(c^2) - 6(a^2)}{9}$$

that b is a multiple of 3 as well, contradicting our assumption that $\gcd(a, b, c) = 1$.

For $n \geq 3$, suppose that $2a^n + 3b^n = 4c^n$. As in the previous case, we can assume $\gcd(a, b, c) = 1$. Since

$$3b^n = 4c^n - 2a^n,$$

b^n must be even. We can then write

$$a^n + 2^{n-1} \cdot 3(b/2)^n = 2c^n,$$

and so a must be even. Then

$$2^{n-1}(a/2)^n + 2^{n-2} \cdot 3(b/2)^n = c^n,$$

and c must be even as well. This contradicts our assumption that $\gcd(a, b, c) = 1$.

Problem 3

Without loss of generality, assume that A and B lie in the first quadrant with $A = (t_1, 1/t_1)$, $B = (t_2, 1/t_2)$, and $t_1 < t_2$. If $P = (t, 1/t)$ with $t_1 \le t \le t_2$, then the area of triangle APB is

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ t_1 & t & t_2 \\ 1/t_1 & 1/t & 1/t_2 \end{vmatrix} = \frac{t_2 - t_1}{2t_1t_2}(t_1 + t_2 - t - t_1t_2/t).$$

When t_1, t_2 are fixed, this is maximized when $t + t_1t_2/t$ is minimized, which by AM-GM exactly holds when $t = \sqrt{t_1t_2}$.
The line AP is given by $y = \frac{t+t_1-x}{tt_1}$, and so the area of the region bounded by the hyperbola and AP is

$$\int_{t_1}^{t} \left(\frac{t + t_1 - x}{tt_1} - \frac{1}{x}\right) dx = \frac{t}{2t_1} - \log\left(\frac{t}{t_1}\right),$$

which at $t = \sqrt{t_1t_2}$ is equal to $\frac{t_2-t_1}{2\sqrt{t_1t_2}} - \log(\sqrt{t_2/t_1})$.
Similarly, the area of the region bounded by the hyperbola and PB is

$$\frac{t_2}{2t} - \frac{1}{2t_2} - \log(\frac{t_2}{t}),$$

which at $t = \sqrt{t_1t_2}$ is

$$\frac{t_2 - t_1}{2\sqrt{t_1t_2}} - \log(\sqrt{t_2/t_1}),$$

as desired.