The Curious Quest

Issue Number 12

Centre for Mathematical Outreach, MAX

"God exists since mathematics is consistent, and the Devil exists since we cannot prove it"

– André Weil

§ Events Board

CMO Student Lecture: Rearranging Infinity

Date: 22–08–2025 **Time:** 3:00 PM
Venue: St. Xavier's College. (Exact venue TBA on `@math.at.xaviers`)
Description: Learn about sequences, series, and a fascinating theorem on a special type of series. Discover how rearranging terms can lead to surprising results!

If you want any pan-Mumbai math/physics/stats event to be featured here, mail us: `centre.math.outreach@gmail.com`

§ Reader's Delight

The Accidental Doctor

Stefan Banach, one of the founding giants of modern functional analysis, had a mind so sharp it often outran academic formalities. According to a famous story, Banach was once strolling through the corridors of the University of Lwów when a colleague stopped him. "There are a few mathematicians in a room who want to discuss some problems with you," he was told. Banach, ever happy to talk math, walked in and began answering their questions with his usual brilliance. Only later did he discover that this was no casual chat, it was his **PhD defense**. Without realizing it, he had just convinced a formal examination committee from Warsaw that he deserved a doctorate.

Figure 1: S. Banach, 1919

It's a delightful tale of pure intellect trumping ceremony, though modern historians suspect the truth was a bit more conventional. Still, whether legend or fact, the story captures Banach perfectly: a mathematician so absorbed in ideas that he might earn a PhD without even knowing he was being examined.

K. Ciesielski & D. Ciesielska, *The Myths of Banach's PhD Thesis*, The Mathematical Intelligencer, 43 (2021).

§ The Problem Arena

Problem 1

Let the functions, $f:(-1,1)\to\mathbb{R}$ and $g:(-1,1)\to[0,1)$ be defined as $f(x)=|2x-1|+|2x+1|$ and $g(x)=x-\lfloor x\rfloor$.
Here $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.
Let $f\circ g:(-1,1)\to\mathbb{R}$ be the composition function function defined by $f\circ g\,(x)=f(g(x))$. Find the number of points in $(-1,1)$ where $f\circ g$ is:
(a) not continuous
(b) not differentiable

Problem 2

Let $g:\mathbb{R}\to\mathbb{R}$ be a continuously differentiable such that $|g'(x)|<1$ for every $x\in\mathbb{R}$. Consider the function $f:\mathbb{R}\to\mathbb{R}$ and $f(x)=g(x)+x$.
(a) Prove that f is injective and f^{-1} is continuous.
(b) Prove that if there exists $k\in[0,1)$ such that $|g'(x)|\le k$ for all $x\in\mathbb{R}$, then f is surjective.

Problem 3

Evaluate the following indefinite integral:

$$\int \sqrt[3]{\tan x}\,dx$$

Hint: Substitute $u^3=\tan x$, and then $t=u^2$.

§ The Enigma Box

Technology, Speed Force, and Magic

After absorbing the powers of The Beyonder, Dr. Doom has transported champions from different universes to his planet, *Battleworld*. There, he hosts the *Tri-Franchise Championship Games*, where three icons face off:

$$A \text{ (Iron Man)},\quad B \text{ (The Flash)},\quad C \text{ (Hermione Granger)}.$$

The competition spans multiple events, from *Quidditch* to the *Wakandan Discus Throw*.
In every event:

- Winner gets x points,
- Runner-up gets y points,
- Third place gets z points,

where $x>y>z>0$ are positive integers.
When the dust (and magical sparks) settled, the scores were:

$$\text{Iron Man: } 22,\quad \text{The Flash: } 9,\quad \text{Hermione: } 9.$$

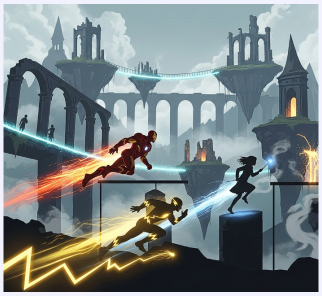

Figure 2: The *TFC*

Naturally, Doom's home franchise took the top spot — pure coincidence, of course. One unforgettable moment came when The Flash won the *400 metres hurdle race* so quickly that even Hermione's Time-Turner could not keep up.

Your challenge:

1. Find the total number of events.
2. Determine who finished second in the *Wakandan Discus Throw* (rumor has it, the result shocked the crowd).

§ This Issue's Contributors

- **Problem 1**: *Omisha Ruhi, St. Xavier's College, Mumbai*
- **Problem 2**: *Aadi Upraity, St. Xavier's College, Mumbai*
- **Problem 3**: *Malhaar Gangan, St. Xavier's College, Mumbai*
- **Enigma Box**: *Daksh Bharadwaj, St. Xavier's College, Mumbai*
- **Reader's Delight**: *Martin John, St. Xavier's College, Mumbai*

We do not claim to be the creators of any questions shared in *The Curious Quest*, unless specified otherwise.

If you have any questions, puzzles, or stories that you want to share, kindly mail them to centre.math.outreach@gmail.com!

§ Hints & Solutions - Previous Issue

Problem 1

Divide the numerator and denominator by n^{12} to obtain:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\right)\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^3\right)\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4\right)}{\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^5\right)^2}$$

By Riemann integration, we see that this is equal to:

$$\frac{\int_0^1 x^2\,dx \int_0^1 x^3\,dx \int_0^1 x^4\,dx}{\left(\int_0^1 x^5\,dx\right)^2} = \boxed{\frac{3}{5}}$$

Problem 2

Drawing the graphs, you get the area required.

Final answer : $\boxed{\frac{11}{32}\text{ sq. units}}$

Problem 3

This is a very famous ISI undergraduate entrance exam question. It's solution is as follows:
(i)

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$f'(1) = \lim_{x \to 1} \frac{e^{x^{10}-1} + (x-1)^2 \sin\left(\frac{1}{x-1}\right) - 1}{x - 1}$$

$$f'(1) = \lim_{x \to 1} \frac{e^{x^{10}-1} - 1}{x - 1} + \lim_{x \to 1} \frac{(x-1)^2 \sin\left(\frac{1}{x-1}\right)}{x - 1}$$

The second limit simplifies to:

$$\lim_{x \to 1} (x-1) \sin\left(\frac{1}{x-1}\right) = 0$$

This is because $\sin\left(\frac{1}{x-1}\right)$ is bounded between -1 and 1, and as $x \to 1$, $(x-1) \to 0$.
For the first limit, we can use the identity $\lim_{y \to 0} \frac{e^y - 1}{y} = 1$. Let $y = x^{10} - 1$. As $x \to 1$, $y \to 0$.

$$\lim_{x \to 1} \frac{e^{x^{10}-1} - 1}{x - 1} = \lim_{x \to 1} \frac{e^{x^{10}-1} - 1}{x^{10} - 1} \cdot \frac{x^{10} - 1}{x - 1}$$

The first part of the product is 1, and the second part is the derivative of x^{10} at $x = 1$, which is $10x^9$ evaluated at $x = 1$, so it is 10. Thus:

$$f'(1) = 1 \cdot 10 + 0 = \boxed{10}$$

(ii) We need to evaluate the limit:

$$\lim_{u \to \infty} \left[100u - u \sum_{k=1}^{100} f\left(\frac{1+k}{u}\right)\right]$$

$$\lim_{u \to \infty} u \left[100 - \sum_{k=1}^{100} f\left(1 + \frac{k}{u}\right)\right]$$

$$\sum_{k=1}^{100} \lim_{u \to \infty} u \left[1 - f\left(1 + \frac{k}{u}\right)\right]$$

Let $h = \frac{k}{u}$. As $u \to \infty$, $h \to 0$. The expression can be rewritten as:

$$\sum_{k=1}^{100} k \lim_{h \to 0} \frac{1 - f(1+h)}{h}$$

$$\sum_{k=1}^{100} k \lim_{h \to 0} \frac{f(1) - f(1+h)}{h} = \sum_{k=1}^{100} k(-f'(1))$$

Since we found $f'(1) = 10$, we have:

$$-f'(1) \sum_{k=1}^{100} k = -10 \times \left(\frac{100 \times 101}{2}\right) = -10 \times 5050 = \boxed{-50500}$$

Ramsey at Recess

We have 5 dots, and every dot is connected to every other dot (so 10 lines total). Each line can be red or blue. We want to count the number of colorings where there is no all-red triangle and no all-blue triangle. The red graph can have at most 6 lines, same for the blue graph. Since together they make all 10 lines, the number of red lines, say x must satisfy: $x \leq 6$ and $10 - x \leq 6$.
So put $x = 4, 5, 6$ and test for those values, final answer should be $\boxed{12}$