The Curious Quest

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"God exists since mathematics is consistent, and the Devil exists since we cannot prove it."

– André Weil

§ Reader's Delight

Henri Poincaré: The Last Universalist of Mathematics

Henri Poincaré was a French Mathematician, He was well known for his contributions in founding topology, which is the study of shapes and spaces, exploring chaos theory before long before it became popular, and coming up with *Poincaré Conjecture*. He had a broad perspective and looked for connections between different areas of mathematics and physics, in addition to problem-solving. For this reason, he has been referred to as the "last universalist" in mathematics.

Figure 1: Henri Poincaré

His reflections on chaos were some of his most unusual. He realized that even seemingly simple systems, like the motion of planets, could display unpredictable behavior. Likewise, one of his greatest pursuits was the question of whether every simply connected three-dimensional space has the structure of a three-dimensional sphere. This problem had resisted solution for nearly a century until Grigori Perelman resolved it in 2003.

Apart from that, in his work on space mechanics which later influenced ideas in modern physics, he researched the motion of planets and stars. His approach, which made even complex concepts seem understandable, won him the admiration of many scientists and students. Poincaré believed that perception was as important as reasoning in solving problems. He proved that mathematics was about more than just numbers; it was also about connections and patterns in the universe itself. His contributions continue to shape modern scientific research and advancements. Although Poincaré lived in the late 19th and early 20th centuries, his ideas continue to shape modern mathematics. His unique blend of creativity and rigorous reasoning established him as one of the most influential mathematicians in history.

"O'Connor, J. J., & Robertson, E. F." "Jules Henri Poincaré (1854-1912) — Biography." *MacTutor History of Mathematics*, School of Mathematics and Statistics, University of St Andrews. .

§ The Problem Arena

Problem 1

Prove that for all positive real numbers a, b, c we have

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}.$$

Problem 2

Prove that at any party with n people, there must exist at least two people who have shaken hands with the same number of other people present.

Problem 3

1. Suppose $f:[1,\infty) \to [0,\infty)$ is continuous. Show that

$$\left(\int_1^\infty f(x)\,dx\right)^2 \leq \int_1^\infty x^2(f(x))^2\,dx$$

2. Further, give a function f such that equality holds when both sides are finite.

Problem 4

Let f be a real function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, $f^{(3)}(x)$ are positive for all x. Suppose that $f^{(3)}(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

Problem 5

Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed Poisson(λ), and let $\bar{X}$ and S^2 denote the sample mean and variance, respectively.

1. Prove that $\bar{X}$ is the best unbiased estimator of λ without using the Cramer-Rao Theorem
2. Prove $\mathbb{E}\left(S^2 \mid \bar{X}\right) = \bar{X}$,and use it to explicitly demonstrate that $V\left(S^2\right) > V\left(\bar{X}\right)$.
3. Using completeness, can a general theorem be formulated for which the identity in part (2) is a special case?

§ The Enigma Box

Chai, Cricket, and Chances

You are a die-hard cricket fan and, miraculously, your IPL team has reached the final series: a best-of-seven playoff to decide the champion. Unfortunately, the opposition is a much stronger side, with a 60% chance of winning any given match against your team.
Sure enough, your team loses the first match of the series, and you are so devastated that you drown your sorrows in endless cups of chai and late-night snacks. When you finally regain your senses, you learn that two more matches have been played.
You rush out and stop the first person you see on the street: "What happened in matches 2 and 3 of the final series?"
"They were split," he replies. "Each team won one match."
Should you be happy?

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If you have any questions, puzzles, or stories that you want to share, kindly mail them to
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§ Hints & Solutions - Previous Issue

Problem 1

x f(x) = $\ln x$
f(x) = $\frac{\ln x}{x}$
Using Leibniz's rule for the n^{th} derivative:

$$f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{x}\right)^{(k)} (\ln x)^{(n-k)} = \left(\sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{1}{x}\right)^{(k)} (\ln x)^{(n-k)} \right) + \binom{n}{n} \left(\frac{1}{x}\right)^{(n)} \ln x$$

Using

$$\left(\frac{1}{x}\right)^{(k)} = (-1)^k \frac{k!}{x^{k+1}} \quad \text{and} \quad (\ln x)^{(n-k)} = (-1)^{n-k-1} \frac{(n-k-1)!}{x^{n-k}},$$

$$f^{(n)}(x) = \left(\sum_{k=0}^{n-1} \binom{n}{k} (-1)^k \frac{k!}{x^{k+1}} \cdot (-1)^{n-k-1} \frac{(n-k-1)!}{x^{n-k}} \right) + \left(\frac{1}{x}\right)^{(n)} \ln x$$

$$= \left(\frac{(-1)^{n-1}}{x^{n+1}} \sum_{k=0}^{n-1} \frac{n!}{k!(n-k)!} k!\,(n-k-1)! \right) + \left(\frac{1}{x}\right)^{(n)} \ln x$$

$$= \left(\frac{(-1)^{n-1}}{x^{n+1}} \sum_{k=0}^{n-1} \frac{n!}{(n-k)} \right) + \left(\frac{1}{x}\right)^{(n)} \ln x$$

$$= \left(\frac{(-1)^{n-1} n!}{x^{n+1}} \sum_{k=0}^{n-1} \frac{1}{(n-k)} \right) + \left(\frac{1}{x}\right)^{(n)} \ln x$$

$$= \left(\frac{(-1)^{n-1} n!}{x^{n+1}} \sum_{j=1}^{n} \frac{1}{j} \right) + \left(\frac{1}{x}\right)^{(n)} \ln x \qquad (\text{where } j = n-k)$$

$$f^{(n)}(1) = (-1)^{n+1} n! \sum_{j=1}^{n} \frac{1}{j} \qquad \left((-1)^{n-1} = (-1)^{n+1},\ \ln 1 = 0\right)$$

$$f^{(n)}(1) = (-1)^{n+1} n! \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

Hence Proved.

Problem 2

Let A be the event that 6 appears exactly once. Let B be the event that 6 appears at least once. We need to find the conditional probability $P(A|B) = \frac{P(A\cap B)}{P(B)}$.

First, we find $P(B)$. The complement of B is the event that no 6 appears in 3 rolls. The probability of not rolling a 6 is $\frac{5}{6}$.

$$P(\text{not } B) = \left(\frac{5}{6}\right)^3$$

Therefore,

$$P(B) = 1 - P(\text{not } B) = 1 - \left(\frac{5}{6}\right)^3$$

Next, we find $P(A \cap B)$. If event A occurs (exactly one 6), then event B (at least one 6) must also have occurred. Thus, $A \cap B = A$.

We calculate $P(A)$ using the binomial probability formula:

$$P(A) = \binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{3-1} = 3 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^2$$

Finally, we can calculate $P(A|B)$:

$$P(A|B) = \frac{P(A)}{P(B)} = \frac{3 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^2}{1 - \left(\frac{5}{6}\right)^3}$$

This corresponds to option (c).

Problem 3

The general n-th term T_n is given by:

$$T_n = \frac{n+1}{\sum_{k=1}^{n}(2k-1)^2}$$

Recall the formula for the sum of the squares of the first n odd numbers:

$$\sum_{k=1}^{n}(2k-1)^2 = \frac{2n(n+1)(2n+1)}{3} = 2n(n+1) + n$$

The sum of the first 50 terms is:

$$S = \sum_{n=1}^{50} \frac{n+1}{\sum_{k=1}^{n}(2k-1)^2}$$

The Postman Workout

The "*greedy algorithm*" for maximizing the postman's distance would have him start at 2, go all the way to 19, then back to 3, then to 17 and so forth, ending at 11. That would result in total distance 17+16+14+12+8+6+4 = **77 units**. Is this the best he can do? The answer is NO.
To find the longest path, the problem is analyzed by looking at the small segments between consecutive addresses in the sorted list (e.g., the distance between house 2 and 3, 3 and 5, etc.). The total distance is the sum of how many times each of these segments is crossed. The central segments can be crossed more times than the outer ones. The maximum theoretical distance is achieved by a specific strategy:

1. **Start and End**: The postman must start at one of the two middle addresses (in this case, 7 or 11) and end at the other.
2. **Zigzag Path**: He must always alternate between an address in the lower half (2, 3, 5, 7) and one in the upper half (11, 13, 17, 19).

An example of an optimal path is $7 \to 19 \to 3 \to 17 \to 5 \to 13 \to 2 \to 11$, ie: **82 units**. This general principle of starting and ending in the middle while zigzagging between the lower and upper halves of the addresses guarantees the longest possible route for any even number of houses.