The Curious Quest

Issue Number 17

Centre for Mathematical Outreach, MAX

"God may not play dice with the universe, but something strange is going on with the prime numbers"

– Paul Erdös

§ Reader's Delight

The Möbius Strip Misstep

Richard Schwartz, a mathematician celebrated for his creativity and curiosity, took on the playful yet deep challenge of finding the shortest closed curve one can draw on a Möbius strip, a surface famous for having only one side and one edge. While its appearance invites wonder, the math behind it had stumped experts for decades. Early in Schwartz's journey, he was tripped up by an almost comical mistake: he was sure that slicing the twisted strip at an angle would give him a parallelogram, when in fact it produces a trapezoid. That simple slip halted his progress and like any good mystery, left the solution tantalizingly out of reach for three years.

Figure 1: Richard Evan Schwartz

Schwartz, known for enjoying simply stated yet tricky problems (and having written math-themed children's books), eventually circled back. With a fresh perspective, he spotted his elementary error. Correcting it allowed him to crack the puzzle and reveal the elegant solution for the Möbius strip's shortest non-contractible curve. His breakthrough not only unlocked a long-standing geometric mystery but also inspired fellow mathematicians, showing that sometimes, even the world's sharpest minds can get tripped by the smallest twists, and that a spirit of fun and persistence can turn even frustration into discovery.

Crowell, Rachel. "Mathematician Solves 50-Year-Old Möbius Strip Puzzle." Scientific American, 12 Sept. 2023, www.scientificamerican.com/article/mathematicians-solve-50-year-old-moebius-strip-puzzle1/

§ The Problem Arena

Problem 1

A philosopher's book club consists of five members: Socrates, Plato, Aristotle, Nietzsche, and Weil. Each month, three members are randomly selected to present a summary of the book.

They now select presenters one at a time until they have selected a complete set of presenters for the month, defined as a group of three that contains at least one Greek philosopher and at least one non-Greek philosopher. What is the probability that the selection process requires exactly three picks?

Problem 2

In probability theory, a sequence of random variables $X_1, X_2, X_3, \cdots$ converges in distribution to a random variable X, shown by $X_n \xrightarrow{\text{d}} X$, if, for the distribution function $F(x)$:

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x), \quad \text{for all } x \text{ at which } F_X(x) \text{ is continuous.}$$

Similarly, such a sequence converges in probability to a random variable X, shown by $X_n \xrightarrow{\text{P}} X$, if:

$$\lim_{n\to\infty} P[|X_n - X| \geq \epsilon] = 0, \text{for all } \epsilon > 0.$$

Does the convergence in distribution of a sequence of random variables imply convergence in probability? Justify.

Problem 3

Evaluate the integral I where:

$$I = \int_0^{\pi} (\sin x + \sin 2x + \sin 3x + \cdots + \sin 2020x)^2 dx$$

§ The Enigma Box

The Father's Choice

A father, mother and son hold a family tournament, playing a two person board game with no ties. The tournament rules are:

- The weakest player chooses the first two contestants.
- The winner of any game plays the next game against the person left out.
- The first person to win two games wins the tournament.

The father is the weakest player, the son the strongest. What starting match should the father choose to maximise his chances of winning the tournament?

§ This Issue's Contributors

- **Problem 1**: *Martin John, St. Xavier's College*
- **Problem 2**: *Isabel Joseph, NMIMS*
- **Problem 3**: *Omisha Ruhi, St. Xavier's College*
- **Enigma Box**: *Daksh Bhardwaj, St. Xavier's College*
- **Reader's Delight**: *Malhaar Gangan, St. Xavier's College*

We do not claim to be the creators of any questions shared in *The Curious Quest*, unless specified otherwise.

If you have any questions, puzzles, or stories that you want to share, kindly mail them to `centre.math.outreach@gmail.com`!

§ Hints & Solutions - Previous Issue

Problem 1

$$\text{We want to prove:} \quad \sum_{\text{cyc}} \frac{1}{a^3+b^3+abc} \le \frac{1}{abc}, \quad a,b,c>0.$$

Consider the algebraic identity $a^3+b^3 \ge a^2b+ab^2, \forall$ positive reals a,b. We show its proof below:

$$a^3+b^3-a^2b-ab^2 = (a^3-a^2b)+(b^3-ab^2) = a^2(a-b)+b^2(b-a)$$

$$= (a-b)(a^2-b^2) = (a-b)(a-b)(a+b) = (a-b)^2(a+b) \ \ge\ 0.$$

$$a^3+b^3 \ \ge\ a^2b+ab^2.$$

$$\implies a^3+b^3+abc \ \ge\ a^2b+ab^2+abc = ab(a+b+c).$$

$$\frac{1}{a^3+b^3+abc} \ \le\ \frac{1}{ab(a+b+c)}.$$

$$\implies \sum_{\text{cyc}} \frac{1}{a^3+b^3+abc} \ \le\ \sum_{\text{cyc}} \frac{1}{ab(a+b+c)}.$$

$$= \frac{1}{a+b+c}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right).$$

$$= \frac{1}{a+b+c}\cdot\frac{a+b+c}{abc}.$$

$$= \frac{1}{abc}.$$

Hence proved.

Problem 2

Consider any person, his or her handshake count must be from $\{0, 1, ..., n-1\}$. If each person gets a unique number, then there will be two people: one with n-1 count and one with 0, since these cannot occur simultaneously, we must exclude atleast one of them. WLOG, let's exclude 0, so we have to choose a number of n people from a set of cardinality $n-1(\{1, 2, .., n-1\})$ and by PHP we are through

Problem 3

1. First, prove that the function $\langle u, v\rangle := \int_1^\infty u(x)v(x)\,dx$ is an inner product on the given space of continuous functions. Then apply the Cauchy-Schwarz inequality on the following functions in the space to obtain the result: $\dfrac{1}{x}$, $x \cdot f(x)$
2. We know that the CS inequality holds only for linearly dependent vectors, so, $xf(x) = \frac{a}{x} \implies f(x) = \frac{a}{x^2}$ for some $a \in \mathbb{R}$.

Problem 4

(based on work by Daniel Stronger) We make repeated use of the following fact: if f is a differentiable function on all of $\mathbb{R}$, $\lim_{x\to-\infty} f(x) \geq 0$, and $f'(x) > 0$ for all $x \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$. (Proof: if $f(y) < 0$ for some y, then $f(x) < f(y)$ for all $x < y$ since $f' > 0$, but then $\lim_{x\to-\infty} f(x) \leq f(y) < 0$.)

From the inequality $f'''(x) \leq f(x)$ we obtain

$$f''(x)f'''(x) \leq f''(x)f(x) < f''(x)f(x) + (f'(x))^2$$

since $f'(x)$ is positive. Applying the fact to the difference between the right and left sides, we get

$$\frac{1}{2}(f''(x))^2 < f(x)f'(x). \tag{1}$$

On the other hand, since $f(x)$ and $f'''(x)$ are both positive for all x, we have

$$2f'(x)f''(x) < 2f'(x)f''(x) + 2f(x)f'''(x).$$

Applying the fact to the difference between the sides yields

$$(f'(x))^2 \leq 2f(x)f''(x). \tag{2}$$

Combining (1) and (2), we obtain

$$\begin{aligned}\frac{1}{2}\left(\frac{(f'(x))^2}{2f(x)}\right)^2 &< \frac{1}{2}(f''(x))^2 \\ &< f(x)f'(x),\end{aligned}$$

or $(f'(x))^3 < 8f(x)^3$. We conclude $f'(x) < 2f(x)$, as desired.

Note: one can actually prove the result with a smaller constant in place of 2, as follows. Adding $\frac{1}{2}f'(x)f'''(x)$ to both sides of (1) and again invoking the original bound $f'''(x) \leq f(x)$, we get

$$\begin{aligned}\frac{1}{2}\left[f'(x)f'''(x) + (f''(x))^2\right] &< f(x)f'(x) + \tfrac{1}{2}f'(x)f'''(x) \\ &\leq \tfrac{3}{2}f(x)f'(x).\end{aligned}$$

Applying the fact again, we get

$$\frac{1}{2}f'(x)f''(x) < \frac{3}{4}f(x)^2.$$

Multiplying both sides by $f'(x)$ and applying the fact once more, we get

$$\frac{1}{6}(f'(x))^3 < \frac{1}{4}f(x)^3.$$

From this we deduce $f'(x) < (3/2)^{1/3}f(x) < 2f(x)$, as desired.

I don't know what the best constant is, except that it is not less than 1 (because $f(x) = e^x$ satisfies the given conditions).

Problem 5

1. Because the Poisson family is an exponential family with $t(x) = x$, $\sum_i X_i$ is a complete sufficient statistic. Any function of $\sum_i X_i$ that is an unbiased estimator of λ is the unique best unbiased estimator of λ.

   Because $\bar{X}$ is a function of $\sum_i X_i$ and $\mathbb{E}[\bar{X}] = \lambda$, $\bar{X}$ is the best unbiased estimator of λ.

2. S^2 is an unbiased estimator of the population variance, that is, $\mathbb{E}[S^2] = \lambda$.

   $\bar{X}$ is a one-to-one function of $\sum_i X_i$. So $\bar{X}$ is also a complete sufficient statistic.

   Thus, $\mathbb{E}[S^2 \mid \bar{X}]$ is an unbiased estimator of λ and, by Lehmann–Scheffé Theorem, it is also the unique best unbiased estimator of λ. Therefore,

   $$\mathbb{E}[S^2 \mid \bar{X}] = \bar{X}.$$

   Then we have

   $$\mathrm{Var}(S^2) = \mathrm{Var}(\mathbb{E}[S^2 \mid \bar{X}]) + \mathbb{E}[\mathrm{Var}(S^2 \mid \bar{X})] = \mathrm{Var}(\bar{X}) + \mathbb{E}[\mathrm{Var}(S^2 \mid \bar{X})],$$

   so

   $$\mathrm{Var}(S^2) > \mathrm{Var}(\bar{X}).$$

3. We formulate a general theorem. Let $T(X)$ be a complete sufficient statistic, and let $T'(X)$ be any statistic other than $T(X)$ such that

   $$\mathbb{E}[T(X)] = \mathbb{E}[T'(X)].$$

   Then,

   $$\mathbb{E}[T'(X) \mid T(X)] = T(X) \quad \text{and} \quad \mathrm{Var}(T'(X)) > \mathrm{Var}(T(X)).$$

Chai, Cricket, and Chances

Setup:

- Opponent win probability per match: $p = 0.6$, so your team win probability: $q = 0.4$.
- Best-of-7 series: first to 4 wins.
- After Game 1: your team lost $\Rightarrow$ score $(0, 1)$.
- After Games 2 and 3: "split" $\Rightarrow$ score $(1, 2)$.

Key Step: Probability of winning the series from state (a, b):

$$\mathbb{P}(\text{win} \mid (a,b)) = \sum_{j=4-a}^{r} \binom{r}{j} q^j (1-q)^{r-j}, \quad r = 7 - (a+b).$$

Case 1: After Game 1 (state $(0, 1)$).

$$r = 6, \;\; \mathbb{P}(\text{win} \mid (0,1)) = \sum_{j=4}^{6} \binom{6}{j} q^j (1-q)^{6-j}.$$

For $q = 0.4$:

$$\mathbb{P}(\text{win} \mid (0,1)) = 0.1792 \;\; (17.92\%).$$

Case 2: After 3 Games (state $(1, 2)$).

$$r = 4, \;\; \mathbb{P}(\text{win} \mid (1,2)) = \binom{4}{3} q^3 (1-q) + \binom{4}{4} q^4.$$

For $q = 0.4$:

$$\mathbb{P}(\text{win} \mid (1,2)) = 0.1792 \;\; (17.92\%).$$

Although it *feels* like splitting Games 2 and 3 should help, the math shows the conditional probability remains unchanged for $q = 0.4$. This is a numerical coincidence.
Your chance of eventually winning the series is still about 17.92%. So, **no, you should not be happier!**